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The twisted photon associated to hyper-Hermitian four-manifolds

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Abstract

The Lax formulation of the hyper-Hermiticity condition in four dimensions is used to derive a pair of potentials that generalises Plebanski's second heavenly equation for hyper-Kähler four-manifolds. A class of examples of hyper-Hermitian metrics which depend on two arbitrary functions of two complex variables is given. The twistor theory of four-dimensional hyper-Hermitian manifolds is formulated as a combination of the Nonlinear Graviton Construction with the Ward transform for anti-self-dual Maxwell fields. © 1999 Elsevier Science B.V. All rights reserved.

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1. Complexified hyper-Hermitian manifolds

A smooth manifold $\mathcal{M}$ equipped with three almost complex structures (I, J, K) satisfying the algebra of quaternions is called hyper-complex iff the almost complex structure

$$\mathcal{J}_\lambda = aI + bJ + cK$$

is integrable for any $(a, b, c) \in S^2$. We shall use a stereographic coordinate $\lambda = (a+ib)/(c-1)$ on S^2 which we will view as a complex projective line $\mathbb{CP}^1$. Let g be a Riemannian metric on $\mathcal{M}$. If $(\mathcal{M}, \mathcal{J}_\lambda)$ is hyper-complex and $g(\mathcal{J}_\lambda X, \mathcal{J}_\lambda Y) = g(X, Y)$ for all vectors X, Y on $\mathcal{M}$ then the triple $(\mathcal{M}, \mathcal{J}_\lambda, g)$ is called a hyper-Hermitian structure. From now on we shall restrict ourselves to oriented four-manifolds. In four dimensions a hyper-complex structure

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defines a conformal structure, which in explicit terms is represented by a conformal frame of vector fields (X, IX, JX, KX), for any $X \in T\mathcal{M}$.

It is well known that this conformal structure is anti-self-dual (ASD) with the orientation determined by the complex structures. Let g be a representative of the conformal structure defined by $\mathcal{J}_\lambda$, and let $\Sigma^{A'B'} = (\Sigma^{00'}, \Sigma^{01'}, \Sigma^{11'})$ be a basis of the space of SD two-forms $\Lambda^2{}_+(\mathcal{M})$ (see Appendix A for notation and conventions). The following holds:

Proposition 1 [1]. *The Riemannian four-manifold $(\mathcal{M}, g)$ is hyper-Hermitian if there exists a one-form A (called a Lee form) depending only on g such that*

$$d\Sigma^{A'B'} = -A \wedge \Sigma^{A'B'}. \tag{1}$$

Moreover if A is exact, then g is conformally hyper-Kähler.

In Section 2 we shall express the hyper-Hermiticity condition on the metric in four dimensions in terms of Lax pairs of vector fields on $\mathcal{M}$. The Lax formulation will be used to encode the hyper-Hermitian geometry in a generalisation of Plebański's formalisms [14]. Some examples of hyper-Hermitian metrics are given in Section 3. In Section 4 we establish the twistor correspondence for the hyper-Hermitian four-manifolds. If $\mathcal{M}$ is real then the associated twistor space is identified with a sphere bundle of almost-complex structures and the resulting twistor theory is well-known [1,13]. We will work with the complexified correspondence and assume that $\mathcal{M}$ is a complex four-manifold. The integrability conditions under which (1) can hold are $dA \in \Lambda^2{}_-(\mathcal{M})$ so dA can formally be identified with an ASD Maxwell field on an ASD background. This will enable us to formulate the twistor theory of hyper-Hermitian manifolds as a nonlinear graviton construction 'coupled' to a Twisted Photon Construction [18].

In Section 5 we make further remarks about the hyper-Hermitian equation, and list some open problems. The spinor notation which is used in the paper is summarised in Appendix A.

2. Hyper-Hermiticity condition as an integrable system

The hyper-Hermiticity condition on a metric g can be reduced to a system of second order PDEs for a pair of functions. [1] The Lax representation for such an equation will be a consequence of the integrability of the twistor distribution. We shall need the following lemma:

Lemma 2. *Let $\nabla_{AA'}$ be four independent holomorphic vector fields on a four-dimensional complex manifold $\mathcal{M}$, and let*

$$L_0 = \nabla_{00'} - \lambda\nabla_{01'}, \qquad L_1 = \nabla_{10'} - \lambda\nabla_{11'}, \quad \textit{where} \quad \lambda \in \mathbb{CP}^1.$$

[1] K.P. Tod has given a generalisation of the first heavenly equation to the case of real hyper-Hermitian four-manifolds.

If

$$[L_0, L_1] = 0 \tag{2}$$

for every λ, then $\nabla_{AA'}$ is a null tetrad for a hyper-Hermitian metric on $\mathcal{M}$. Every hyper-Hermitian metric arises in this way.

Proof. We use the spinor notation of Penrose and Rindler [12]. Let $\nabla_{AA'}$ be a tetrad of holomorphic vector fields on $\mathcal{M}$. A central result of twistor theory [9,11] (see also Section 4 of this paper) is that $\nabla_{AA'}$ determines an anti-self-dual conformal structure if and only if the distribution on the primed-spin bundle $S^{A'}$ spanned by the vectors

$$L_A = \pi^{A'}\nabla_{AA'} + \Gamma_{AA'B'C'}\pi^{A'}\pi^{B'}\frac{\partial}{\partial \pi_{C'}}$$

is integrable. This then implies that the spin bundle is foliated by the horizontal lifts of α-surfaces. Here $\pi^{A'} = \pi^{0'}o^{A'} + \pi^{1'}\iota^{A'}$ is the spinor determining an α-surface and is related to $\lambda = (-\pi^{1'}/\pi^{0'})$. From the general formula

$$d\Sigma^{A'B'} + 2\Gamma_{C'}^{(A'} \wedge \Sigma^{B')C'} = 0,$$

we conclude that $\Gamma_{AA'B'C'} = -A_{A(C'}\varepsilon_{B')A'}$ for some $A_{AA'}$ and

$$L_A = \pi^{A'}\nabla_{AA'} + (1/2)\pi^{A'}A_{AA'}\Upsilon,$$

where $\Upsilon = \pi^{A'}/\partial\pi^{A'}$ is the Euler vector field. We have

$$\begin{aligned}
[L_A, L_B] &= \pi^{A'}\pi^{B'}([\nabla_{AA'}, \nabla_{BB'}] + 1/2([\nabla_{BB'}, A_{AA'}\Upsilon] - [\nabla_{AA'}, A_{BB'}\Upsilon]))\\
&= \pi^{A'}\pi^{B'}([\nabla_{AA'}, \nabla_{BB'}] + (1/2)\varepsilon_{AB}\nabla^{C}{}_{(A'}A_{B')C}\Upsilon)\\
&= \pi^{A'}\pi^{B'}[\nabla_{AA'}, \nabla_{BB'}] \qquad \text{since } dA \text{ is ASD.}
\end{aligned} \tag{3}$$

We shall introduce the rotation coefficient C_{ab}^{c} defined by

$$[\nabla_a, \nabla_b] = C_{ab}^{c}\nabla_c,$$

They satisfy $C_{abc} = \Gamma_{acb} - \Gamma_{bca}$. From the last formula we can find a spinor decomposition of C_{abc},

$$C_{abc} = C_{ABCC'}\varepsilon_{A'B'} + C_{A'B'CC'}\varepsilon_{AB},$$

where

$$C_{A'B'CC'} = \Gamma_{C(A'B')C'} + \varepsilon_{C'(B'}\Gamma_{A')AC}{}^{A}. \tag{4}$$

Collecting (3), and (4) we obtain

$$[L_A, L_B] = \varepsilon_{AB}\pi^{A'}\pi^{B'}((1/2)A^{C}_{B'}\varepsilon_{A'}{}^{C'} + \varepsilon_{A'}{}^{C'}\Gamma_{B'D}{}^{CD})\nabla_{CC'}.$$

We choose a spin frame (o^A, ι^A) constructed from two independent solutions to the charged neutrino equation

$$(\nabla_{AA'} + (1/2)A_{AA'})o^A = (\nabla_{AA'} + (1/2)A_{AA'})\iota^A = 0.$$

In this frame $\Gamma_{AA'}{}^{BA} = -(1/2)A^{B}_{A'}$. To obtain Eq. (2) we project L_A to the projective prime-spin bundle $\mathcal{F} = \mathbb{P}S_{A'}$. In terms of the tetrad

$$[\nabla_{A0'}, \nabla_{B0'}] = 0, \tag{5}$$

$$[\nabla_{A0'}, \nabla_{B1'}] + [\nabla_{A1'}, \nabla_{B0'}] = 0, \tag{6}$$

$$[\nabla_{A1'}, \nabla_{B1'}] = 0. \tag{7}$$

The formulation of the hyper-complex condition in formulae (5)–(7) was in the Riemannian case given in [8] and used in [7]. The Lax equation (2) can be interpreted as the anti-self-dual Yang–Mills equations on $\mathbb{C}^4$ with the gauge group G =Diff($\mathcal{M}$), reduced by four translations in $\mathbb{C}^4$. □

Define (1, 1) tensors $\mathcal{J}_{B'}^{A'} := e^{AA'} \otimes \nabla_{AB'}$. As a consequence of (5)–(7) the Nijenhuis tensors

$$\begin{aligned} N_{B'}^{A'}(X, Y) := {} & (\mathcal{J}_{B'}^{A'})^2[X, Y] - \mathcal{J}_{B'}^{A'}[\mathcal{J}_{B'}^{A'}X, Y] \\ & -\mathcal{J}_{B'}^{A'}[X, \mathcal{J}_{B'}^{A'}Y] + [\mathcal{J}_{B'}^{A'}X, \mathcal{J}_{B'}^{A'}Y] \end{aligned} \tag{8}$$

vanish for arbitrary vectors X and Y. Tensors $\mathcal{J}_{A'}^{B'}$ can be treated as 'complexified complex structures' on $\mathcal{M}$. The complex structure $\mathcal{J}_\lambda$ on $S^{A'}$ can be conveniently expressed as

$$\mathcal{J}_\lambda = \pi_{A'}\tilde{\pi}^{B'}\mathcal{J}_{B'}^{A'}, \quad \text{where } \pi_{A'}\tilde{\pi}^{A'} = 1.$$

Now we shall fix some remaining gauge and coordinate freedom. Eqs. (5)–(7) will be reduced to a coupled system of nonlinear differential equations for a pair of functions.

Proposition 3. *Let $x^{AA'} = (x^A, w^A)$ be local null coordinates on $\mathcal{M}$ and let Θ^A be a pair of complex valued functions on $\mathcal{M}$ which satisfy*

$$\frac{\partial^2 \Theta_C}{\partial x_A \partial w^A} + \frac{\partial \Theta_B}{\partial x^A}\frac{\partial^2 \Theta_C}{\partial x_A \partial x_B} = 0. \tag{9}$$

Then

$$ds^2 = dx_A \otimes dw^A + \frac{\partial \Theta_A}{\partial x^B} dw^B \otimes dw^A \tag{10}$$

is a hyper-Hermitian metric on $\mathcal{M}$. Conversely every hyper-Hermitian metric locally arises by this construction.

Eq. (1) and its connection with a scalar form of (9) was investigated by different methods in [6] in the context of weak heavenly spaces. Other integrable equations associated to hyper-Hermitian manifolds have been studied in [4].

Proof. Choose a conformal factor such that $A_{AA'} = o_{A'}A_A$ for some $o_{A'}$ and A_A. This can be done since the two-form $\Sigma^{1'1'}$ is simple and therefore Eq. (1) together with the Frobenius theorem imply the existence of the conformal factor such that $d\Sigma^{1'1'} = 0$. Hence, using Darboux's theorem, one can introduce canonical coordinates w^A such that

$$\Sigma^{1'1'} = (1/2)\varepsilon_{AB} dw^A \wedge dw^B,$$

and choose an unprimed-spin frame so that $o_{A'}e^{AA'} = dw^A$. Coordinates w^A parametrise the space of null surfaces tangent to $o^{A'}$, i.e. $o^{A'}\nabla_{AA'}w^B = 0$. Consider

$$\mathcal{J}_{0'}^{1'} = o^{B'} dw^A \otimes \nabla_{AB'}.$$

The tensor $\mathcal{J}_{0'}^{1'}$ is a degenerate complex structure. Therefore $(\mathcal{J}_{0'}^{1'})^2 = 0$ where $\mathcal{J}_{0'}^{1'}$ is now thought of as a differential operator acting on forms. Let h be a function on $\mathcal{M}$. Then

$$\mathcal{J}_{0'}^{1'} \lrcorner\, d(\mathcal{J}_{0'}^{1'}(dh)) = 0 \quad \text{implies that} \quad [\nabla_{A0'}, \nabla_{B0'}] = 0,$$

and our choice of the spin frame is consistent with (5)–(7). By applying the Frobenius theorem we can find coordinates x^A such that

$$\nabla_{A0'} = \frac{\partial}{\partial x^A}, \qquad \nabla_{A1'} = \frac{\partial}{\partial w^A} - \Theta_A{}^B \frac{\partial}{\partial x^B}.$$

Using Eq. (6), we deduce the existence of a potential Θ_A such that $\Theta_A{}^B = \nabla_{A0'}\Theta^B$. Now (7) gives the field equations (9)

$$\frac{\partial^2 \Theta_C}{\partial x_A \partial w^A} + \frac{\partial \Theta_B}{\partial x^A}\frac{\partial^2 \Theta_C}{\partial x_A \partial x_B} = 0.$$

The dual frame is

$$e^{A0'} = dx^A + \frac{\partial \Theta^A}{\partial x^B} dw^B, \qquad e^{A1'} = dw^A,$$

which justifies formula (10). □

In the adopted gauge, the Maxwell potential is

$$A = \frac{\partial^2 \Theta^B}{\partial x^A \partial x^B} dw^A$$

and $\nabla^a A_a = 0$, i.e. this is a 'Gauduchon gauge'. Electromagnetic gauge transformations on A correspond to conformal rescalings of the tetrad (which preserve the hypercomplex structure). The second heavenly equation (and therefore the hyper-Kähler condition) follows from (9) if in addition $\nabla_{A0'}\Theta^A = 0$. This condition guarantees the existence of a scalar function Θ which satisfies the second Plebański equation

$$\frac{\partial^2 \Theta}{\partial w^A \partial x_A} + \frac{1}{2}\frac{\partial^2 \Theta}{\partial x^B \partial x^A}\frac{\partial^2 \Theta}{\partial x_B \partial x_A} = 0$$

such that $\Theta^A = \nabla^A{}_{0'}\Theta$. In this case A is exact so can be gauged away by a conformal rescaling.

3. Examples

We look for solutions to (9) for which the linear and nonlinear terms vanish separately, i.e.

$$\frac{\partial^2 \Theta_C}{\partial x_A \partial w^A} = \frac{\partial \Theta_B}{\partial x^A}\frac{\partial^2 \Theta_C}{\partial x_A \partial x_B} = 0. \tag{11}$$

Put $w^A = (w, z), x^A = (y, -x)$. A simple class of solutions to (11) is provided by

$$\Theta_0 = ax^l, \quad \Theta_1 = by^k, \qquad k, l \in \mathbb{Z}, \quad a, b \in \mathbb{C}.$$

The corresponding metric and the Lee form are

$$\begin{aligned} ds^2 &= dw \otimes dx + dz \otimes dy + (alx^{l-1} + bky^{k-1}) dw \otimes dz, \\ A &= b(k-1)ky^{k-2} dw - a(l-1)lx^{l-1} dz. \end{aligned} \tag{12}$$

From calculating the invariant

$$C_{ABCD}C^{ABCD} = (3/2)abk(k-1)(k-2)l(l-1)(l-2)x^{l-3}y^{k-3},$$

we conclude that the metric (12) is in general of type I or D (or type III or N if a or b vanish, or $k < 3$ or $l < 3$).

3.1. Hyper-Hermitian elementary states

A more interesting class of solutions (which generalise the metric of Sparling and Tod [17] to the hyper-Hermitian case) is given by

$$\Theta_C = \frac{1}{x_A w^A} F_C(W^A), \tag{13}$$

where $W^A = w^A/(x_B w^B)$ and F_C are two arbitrary complex functions of two complex variables. The corresponding metric is

$$ds^2 = dx_A \otimes dw^A + \frac{1}{(x_A w^A)^2}\left(F_C + \frac{w^B}{(x_A w^A)}\frac{\partial F_C}{\partial W^B}\right) dw^C \otimes (w_A dw^A).$$

This metric is singular at the light-cone of the origin. The singularity may be moved to infinity if we introduce new coordinates $X^A = x^A/(x_B w^B)$, $W^A = w^A/(x_B w^B)$ and rescale the metric by $(X_A W^A)^2$. This yields

$$\begin{aligned} \hat{ds^2} = {} & dX_A \otimes dW^A + \left(F_B + W^C \frac{\partial F_B}{\partial W^C}\right) \\ & \times((X_A W^A) dW^B - W^B d(X_A W^A)) \otimes (W_A dW^A) \end{aligned} \tag{14}$$

and

$$A = -\left(3W^A F_A + 5W^A W^B \frac{\partial F_A}{\partial W^B} + W^A W^B W^C \frac{\partial^2 F_A}{\partial W^B \partial W^C}\right) W_D dW^D.$$

The metric of Sparling and Tod corresponds to setting $F_A = W_A$.

Let us consider the particular case $F_A = (aW^kZ^l, bW^mZ^n)$. The metric is

$$ds^2 = dw \otimes dx + dz \otimes dy + \left(\frac{a(k+l+1)w^kz^l}{(wx+zy)^{k+l+2}}dw + \frac{b(m+n+1)w^mz^n}{(wx+zy)^{m+n+2}}dz\right) \otimes (wdz - zdw). \tag{15}$$

If $a = -b$, $l = n+1$, $k = m-1$ then $\Theta_A = \nabla_{A0'}\Theta$ where $\Theta = -aw^kz^{l-1}(wx+zy)^{-(k+l)}$. For these values of parameters the metric is hyper-Kähler and of type N.

Some solutions to (11) have real Euclidean slices. For example

$$\Theta_0 = -\frac{y\,(2\,wx + zy)}{w^2\,(wx+zy)^2}, \quad \Theta_1 = -\frac{y^2}{w\,(wx+zy)^2}$$

with $w = \bar{x}$, $z = \bar{y}$ yield a solution of type D, which is conformal to the Eguchi–Hanson metric.

4. The twistor construction

In this section we shall establish the correspondence between complexified hyper-Hermitian four-manifolds and three-dimensional twistor spaces with additional structures. We shall also look at examples given in Section 3 from the twistor point of view. We begin with recalling basic facts about the twistor correspondences for ASD spaces [9,11].

Define α-surfaces as null self-dual two-dimensional surfaces in $\mathcal{M}$. The correspondence space $\mathcal{F}$ is a set of pairs (x, λ) where $x \in \mathcal{M}$ and $\lambda \in \mathbb{CP}^1$ parametrises α-surfaces through x in $\mathcal{M}$. We represent $\mathcal{F}$ as the quotient of the primed-spin bundle $S^{A'}$ with fibre coordinates $\pi_{A'}$ by the Euler vector field $\pi^{A'}/\partial\pi^{A'}$ so that the fibre coordinates are related by $\lambda = \pi_{0'}/\pi_{1'}$. The space $\mathcal{F}$ possesses a natural two-dimensional distribution (called the twistor distribution, or the Lax pair, to emphasise the analogy with integrable systems). The Lax pair on $\mathcal{F}$ arises as the image under the projection $TS^{A'} \longrightarrow T\mathcal{F}$ of the distribution spanned by

$$\pi^{A'}\nabla_{AA'} + \Gamma_{AA'B'C'}\pi^{A'}\pi^{B'}\frac{\partial}{\partial\pi_{C'}}$$

and is given by

$$L_A = (\pi_{1'}^{-1})(\pi^{A'}\nabla_{AA'} + f_A\partial_\lambda), \quad \text{where } f_A = (\pi_{1'}^{-2})\Gamma_{AA'B'C'}\pi^{A'}\pi^{B'}\pi^{C'}. \tag{16}$$

The integrability of the twistor distribution is equivalent to $C_{A'B'C'D'} = 0$, the vanishing of the self-dual Weyl spinor. The twistor space arises as a factor space of $\mathcal{F}$ by the twistor distribution. This leads to a double fibration

$$\mathcal{M} \xleftarrow{p} \mathcal{F} \xrightarrow{q} \mathcal{PT}. \tag{17}$$

The existence of L_A can also be deduced directly from the correspondence with $\mathcal{PT}$. The basic twistor correspondence [11] states that points in $\mathcal{M}$ correspond in $\mathcal{PT}$ to rational curves with normal bundle $\mathcal{O}^A(1) = \mathcal{O}(1) \oplus \mathcal{O}(1)$. Let l_x be the line in $\mathcal{PT}$ that corresponds to $x \in \mathcal{M}$. The normal bundle to l_x consists of vectors tangent to x (horizontally lifted to $T_{(x,\lambda)}\mathcal{F}$) modulo the twistor distribution. Therefore we have a sequence of sheaves over $\mathbb{C}P^1$

$$0 \longrightarrow D \longrightarrow \mathbb{C}^4 \longrightarrow \mathcal{O}^A(1) \longrightarrow 0.$$

The map $\mathbb{C}^4 \longrightarrow \mathcal{O}^A(1)$ is given by $V^{AA'} \longrightarrow V^{AA'}\pi_{A'}$. Its kernel consists of vectors of the form $\pi^{A'}\lambda^A$ with λ^A varying. The twistor distribution is therefore $D = \mathcal{O}(-1) \otimes S^A$ and so L_A, the global section of $\Gamma(D \otimes \mathcal{O}(1) \otimes S_A)$, has the form (16).

We have

Proposition 4. *Let $\mathcal{PT}$ be a three-dimensional complex manifold with the following structures:*
(A) *a projection $\mu : \mathcal{PT} \longrightarrow \mathbb{CP}^1$,*
(B) *a four-complex dimensional family of sections with a normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$.*
Then the moduli space $\mathcal{M}$ of sections of μ is equipped with hyper-Hermitian structure. Conversely, given a hyper-Hermitian four-manifold there will always exist a corresponding twistor space satisfying conditions (A) and (B).

Remarks 1.
(i) *The integrability conditions under which (1) can hold are $dA \in \Lambda^2{}_-(\mathcal{M})$ so dA can be identified with an ASD Maxwell field on an ASD background. This suggests that hyper-Hermitian manifolds can be studied with respect to the Twisted Photon Construction [18], associated with dA. Let $K = \Lambda^3(\mathcal{PT})$ be the canonical line bundle. Proposition 4 is different from the original Nonlinear Graviton construction because the line bundle $L := K^* \otimes \mathcal{O}(-4)$, where $\mathcal{O}(-4) = \mu^*(T^*\mathbb{CP}^1)^2$, is in general nontrivial over $\mathcal{PT}$. It is the twisted photon line bundle associated with dA.*
(ii) *If $\mathcal{M}$ is compact then it follows from Hodge theory that $dA = 0$ and the hyper-Hermitian structure is locally conformally hyper-Kähler. We focus on the noncompact case.*
(iii) *If $\mathcal{M}$ is real then $\mathcal{PT}$ is equipped with an anti-holomorphic involution preserving (A) and we recover a result closely related to one of Pedersen and Swann* [13] *who constructed a twistor space corresponding to a real four-dimensional ASD Einstein–Weyl metric with vanishing scalar curvature.*
(iv) *The correspondence is preserved under holomorphic deformations of $\mathcal{PT}$ which preserve (A).*

Proof. Let $\lambda = \pi_{0'}/\pi_{1'}$ be an affine coordinate on $\mathbb{CP}^1$. $\mathcal{PT}$ can be covered by two sets, U and $\tilde{U}$ with $|\lambda| < 1 + \epsilon$ on U and $|\lambda| > 1 - \epsilon$ on $\tilde{U}$ with (ω^A, λ) being the coordinates on U and $(\tilde{\omega}^A, \lambda^{-1})$ on $\tilde{U}$. The twistor space $\mathcal{PT}$ is then determined by the transition function $\tilde{\omega}^B = \tilde{\omega}^B(\omega^A, \pi_{A'})$ on $U \cap \tilde{U}$. Let l_x be the line in $\mathcal{PT}$ that corresponds to $x \in \mathcal{M}$ and let $Z \in \mathcal{PT}$ lie on l_x. We denote by $\mathcal{F}$ the correspondence space $\mathcal{PT} \times \mathcal{M}|_{Z \in l_x} = \mathcal{M} \times \mathbb{CP}^1$ and use the double fibration picture (17).

Consider the line bundle

$$L = K^* \otimes \mathcal{O}(-4)$$

over $\mathcal{PT}$ given by the transition function $f = \det(\partial\tilde{\omega}^A/\partial\omega^B)$. When pulled back to $\mathcal{F}$ it satisfies

$$L_A f = 0.$$

Since $H^1(\mathcal{F}, \mathcal{O}) = 0$, we can perform the splitting $f = h_0 h_\infty^{-1}$. By the standard Liouville arguments (see [18]) we deduce that

$$h_0^{-1} L_A(h_0) = h_\infty^{-1} L_A(h_\infty) = -(1/2)A_A, \tag{18}$$

where $A_A = A_{AB'}\pi^{B'}$ is global on $\mathcal{F}$. The integrability conditions imply that $F_{AB} = \nabla_{A'(A} A^{A'}_{B)}$ is an ASD Maxwell field on the ASD background. The one-form $A = A_{AA'}e^{AA'}$ is a Maxwell potential. The canonical line bundle of $\mathcal{PT}$ is $K = \mathcal{O}(-4) \otimes L^*$. To obtain a global, line bundle valued three-form on $\mathcal{PT}$ one must tensor the last equation with $\mathcal{O}(4) \otimes L$. We pick a global section $\xi \in \Gamma(K \otimes \mathcal{O}(4) \otimes L)$ and restrict ξ to l

$$\xi|_l = \Sigma_\lambda \wedge \pi_{A'} d\pi^{A'}, \tag{19}$$

where $\pi_{A'} d\pi^{A'} \in \Omega^1 \otimes \mathcal{O}(2)$. A two-form

$$\Sigma_\lambda \in \Gamma(\Lambda^2(\mu^{-1}(\lambda)) \otimes \mathcal{O}(2) \otimes L) \tag{20}$$

is defined on vectors vertical with respect to μ by $\Sigma_\lambda(U, V)\pi_{A'} d\pi^{A'} = \xi(U, V, \ldots)$. Let $p^*\Sigma_\lambda$ be the pullback of Σ_λ to $\mathcal{F}$. Note that if

$$A \longrightarrow A - d\phi \ \text{(gauge transformation on } L) \qquad \text{then } p^*\Sigma_\lambda \longrightarrow e^{\phi} p^*\Sigma_\lambda.$$

Let $p^*\Sigma_\lambda$ be defined over U and $p^*\widetilde{\Sigma}_\lambda$ over $\tilde{U}$. We have $f(p^*\Sigma_\lambda) = p^*\widetilde{\Sigma}_\lambda$. By definition, $p^*\Sigma_\lambda$ descends to the twistor space, i.e.,

$$\mathcal{L}_{L_A}(p^*\Sigma_\lambda) = 0. \tag{21}$$

We make use of the splitting formula, and define (on $\mathcal{F}$) $\Sigma_0 = h_0(p^*\Sigma_\lambda)$. The line bundle valued two-form Σ_0 is a globally defined object on $\mathcal{F}$, and therefore it is equal to $\pi_{A'}\pi_{B'}\Sigma^{A'B'}$. Note that Σ_0 does not descend to $\mathcal{PT}$. Fix $\lambda \in \mathbb{CP}^1$ (which gives a copy $\mathcal{M}_\lambda$ of $\mathcal{M}$ in $\mathcal{F}$) and apply (21). This yields

$$\mathcal{L}_{L_A}\Sigma_0 = h_0^{-1} L_A(h_0)\Sigma_0.$$

After some work we obtain formula (1):

$$d\Sigma^{A'B'} = -A \wedge \Sigma^{A'B'}. \tag{22}$$

The integrability conditions for the last equation are guaranteed by the existence of solutions to (18). Eq. (22) and the forward part of Proposition 1 imply that $\mathcal{M}$ is equipped with hyper-Hermitian structure. If the line bundle L over $\mathcal{PT}$ is trivial, then $\mathcal{M}$ is conformally hyper-Kähler.

Now we discuss the converse problem of recovering various structures on $\mathcal{PT}$ from the geometry of $\mathcal{M}$. Let $\mathcal{M}$ be a hyper-Hermitian four-manifold. Therefore $C_{A'B'C'D'} = 0$ and there exists a twistor space satisfying condition (A). Eq. (22) implies that $F = dA$ is an ASD Maxwell field, and we can solve

$$\pi^{A'}(\nabla_{AA'} + (1/2)A_{AA'})\rho = 0$$

on each α-surface (self-dual, two-dimensional null surface in $\mathcal{M}$). We define fibres of L as one-dimensional spaces of solutions to the last equation. The solutions on α-surfaces intersecting at $p \in \mathcal{M}$ can be compared at one point, so L restricted to a line l_x in $\mathcal{PT}$ is trivial. In order to prove that $\mathcal{PT}$ is fibred over $\mathbb{CP}^1$ notice that equation $\pi^{A'}(\nabla_{AA'} + (1/2)A_{AA'})\pi_{B'} = 0$ implies $\pi^{A'}\nabla_{AA'}\lambda = 0$, so λ and $1/\lambda$ descend to give meromorphic functions on twistor space and defines the map $\mathcal{PT} \to \mathbb{CP}^1$. □

4.1. Examples

In this section we shall give the twistor correspondence for the family of hyper-Hermitian metrics (15) found in Section 3. First we shall look at the passive twistor constructions of Θ_C by the contour integral formulae. It will turn out that Θ_C are examples of Penrose's elementary states. Then we explain how the cohomology classes corresponding to Θ_C can be used to deform a patching description of $\mathcal{PT}$. The deformed twistor space will, by Proposition 4, give rise to the metric (15). Both passive and active constructions in this section use methods developed by Sparling in his twistorial treatment of the Sparling–Tod metric.

Parametrise a section of $\mu : \mathcal{PT} \longrightarrow \mathbb{CP}^1$ by the coordinates

$$x^{AA'} := \left.\frac{\partial \omega^A}{\partial \pi_{A'}}\right|_{\pi_{A'}=o_{A'}} = \begin{pmatrix} y & w \\ -x & z \end{pmatrix},$$

so that

$$x^{A1'} = w^A = (w, z), \quad x^{A0'} = x^A = (y, -x).$$

Let us consider the particular case $F_A = (aW^kZ^l, bW^mZ^n)$ discussed in Section 3.1. We work on the nondeformed twistor space $\mathcal{PT}$ with homogeneous coordinates $(\omega^A, \pi_{A'})$. On the primed-spin bundle $\omega^0 = \pi_{1'}(w + \lambda y)$, $\omega^1 = \pi_{1'}(z - \lambda x)$. Consider two twistor functions (sections of $H^1(\mathbb{CP}^1, \mathcal{O}(-2))$

$$h_0 = (-1)^k a \frac{(\pi_{0'})^{k+l}}{(\omega^0)^{l+1}(\omega^1)^{k+1}}, \quad h_1 = (-1)^m b \frac{(\pi_{0'})^{m+n}}{(\omega^0)^{n+1}(\omega^1)^{m+1}},$$

where $a, b \in \mathbb{C}$ and $k, l, m, n \in \mathbb{Z}$ are constant parameters. Then

$$\Theta_A(w, z, x, y) = \frac{1}{2\pi i}\oint_\Gamma h_A(\omega^B, \pi_{B'})\pi_{A'}d\pi^{A'}.$$

Here Γ is a contour in l_x, the $\mathbb{CP}^1$ that corresponds to $(w, z, x, y) \in \mathcal{M}$. It separates the two poles of the integrand. To find Θ^A we compute the residue at one of these poles, which gives

$$\Theta_0 = a\frac{w^k z^l}{(wx + zy)^{k+l+1}}, \quad \Theta_1 = b\frac{w^m z^n}{(wx + zy)^{m+n+1}}, \tag{23}$$

and hence the metric (15).

Now we shall use h_A to deform the complex structure of $\mathcal{PT}$. We change the standard patching relations by setting

$$\tilde{\omega}^A = f^A(\omega^A, t),$$

where t is a deformation parameter and f^A is determined by the deformation equations

$$\frac{\mathrm{d}f^0}{\mathrm{d}t} = \frac{b\pi_{0'}^{m+n+3}}{(\tilde{\omega}^0)^{n+1}(\tilde{\omega}^1)^{m+1}}(-1)^m, \quad \frac{\mathrm{d}f^1}{\mathrm{d}t} = \frac{a\pi_{0'}^{k+l+3}}{(\tilde{\omega}^0)^{l+1}(\tilde{\omega}^1)^{k+1}}(-1)^{k+1}. \tag{24}$$

This equation has a first integral. If $a = -b, l = n + 1, k = m - 1$ then (24) implies that $\omega^0\omega^1 = \tilde{\omega}^0\tilde{\omega}^1$ is a global twistor function. When pulled back to the spin bundle this can be expressed as $P_{A'B'}\pi^{A'}\pi^{B'}$, and the corresponding metric admits a null Killing vector $K_{AA'}$ given by

$$\nabla_{AC'}P_{A'B'} = K_{A(A'}\varepsilon_{B')C'}.$$

Assume that $n + 1 \neq l$, and $k + 1 \neq m$. Then the first integral of (24)

$$Q = \frac{a(\pi^{0'})^{k+l+3}(-1)^{k+1}}{n+1-l}(\omega^0)^{n+1-l} + \frac{b(\pi^{0'})^{m+n+3}(-1)^{m+1}}{k+1-m}(\omega^1)^{k+1-m}$$

is given by a function homogeneous of degree $k + n + 4$. Its pullbacks to $\mathcal{F}$ (which we also denote Q) satisfies $L_A(Q) = 0$. This implies the existence of a Killing spinor of valence $(0, k + n + 4)$ on $\mathcal{M}$.

5. Further remarks

5.1. Symmetries

Eq. (9) has the obvious first integral given by functions Λ_C which satisfy

$$\frac{\partial\Theta_C}{\partial w^A} + \frac{\partial\Theta_B}{\partial x^A}\frac{\partial\Theta_C}{\partial x^B} = \frac{\partial\Lambda_C}{\partial x^A}.$$

It is implicit from the twistor construction that Eq. (9) has infinitely many first integrals given by hidden symmetries. These will be studied (and the associated hierarchy of equations [5]) in a subsequent paper. Here we give a description of those symmetries that correspond to the pure gauge transformations.

Let M be a vector field on $\mathcal{M}$. Define $\delta^0_M \nabla_{AA'} := [M, \nabla_{AA'}]$. This is a pure gauge transformation corresponding to the addition of $\mathcal{L}_M g$ to the space–time metric.

Once a coordinate system leading to Eq. (9) has been selected, the field equations will not be invariant under all the diff($\mathcal{M}$) transformations. We restrict ourselves to transformations that preserve the canonical structures on $\mathcal{M}$, namely

$$\Sigma^{1'1'} = (1/2) dw_A \wedge dw^A, \quad \text{and} \quad \mathcal{J}^{1'}_{0'} = dw^A \otimes \frac{\partial}{\partial x^A}.$$

The condition $\mathcal{L}_M \Sigma^{0'0'} = \mathcal{L}_M \mathcal{J}^{1'}_{0'} = 0$ implies that M is given by

$$M = \frac{\partial h}{\partial w_A} \frac{\partial}{\partial w^A} + \left(g^A - x^B \frac{\partial^2 h}{\partial w_A \partial w^B} \right) \frac{\partial}{\partial x^A},$$

where $h = h(w^A)$ and $g^A = g^A(w^B)$. Space–time is now viewed as a tangent bundle $\mathcal{M} = T\mathcal{N}^2$ with w^A being coordinates on the two-dimensional complex manifold $\mathcal{N}^2$. The full diff($\mathcal{M}$) symmetry breaks down to sdiff($\mathcal{N}^2$) which acts on $\mathcal{M}$ by Lie lift. Let $\delta^0_M \Theta$ corresponds to $\delta^0_M \nabla_{AA'}$ by

$$\delta^0_M \nabla_{A1'} = \frac{\partial \delta^0_M \Theta^B}{\partial x^A} \frac{\partial}{\partial x^B}.$$

The 'pure gauge' elements are

$$\delta^0_M \Theta^B = \mathcal{L}_M(\Theta^B) + F^B - x^A \frac{\partial g^B}{\partial w^A} + x^A x^C \frac{\partial^2 h}{\partial w^A \partial w^C \partial w_B},$$

where F^B, g^A and h are functions of w^B only.

5.2. *$gl(2, \mathbb{C})$ connection*

A natural connection which arises in hyper-Hermitian geometry is the Obata connection [10]. In this section we discuss other possible choices of connections associated with hyper-Hermitian geometry. We shall motivate our choices by considering the conformal rescalings of the null tetrad. The first Cartan structure equations are

$$de^{AA'} = e^{BA'} \wedge \Gamma^A{}_B + e^{AB'} \wedge \Gamma^{A'}{}_{B'}.$$

Rescaling $e^{AA'} \longrightarrow \hat{e}^{AA'} = e^{\phi} e^{AA'}$ yields

$$d\hat{e}^{AA'} = \hat{e}^{BA'} \wedge \Gamma^A{}_B + \hat{e}^{AB'} \wedge \Gamma^{A'}{}_{B'} + d\phi \wedge \hat{e}^{AA'}.$$

The last equation can be interpreted in (at least) three different ways;

(a) Introduce the torsion three-form by $T = *(d\phi) = T_{abc} \hat{e}^a \wedge \hat{e}^b \wedge \hat{e}^c$. Then

$$d\hat{e}^a + \Gamma^a{}_b \wedge \hat{e}^b = T^a,$$

where $T^a = (1/2) T^a_{bc} \hat{e}^b \wedge \hat{e}^c$.

(b) Use the torsion-free $sl(2, \mathbb{C}) \oplus \widetilde{sl}(2, \mathbb{C})$ spin connection

$$\Gamma_{AB} \longrightarrow \Gamma_{AB} + 1/4 * (d\phi \wedge \Sigma_{AB}), \quad \Gamma_{A'B'} \longrightarrow \Gamma_{A'B'} + 1/4 * (d\phi \wedge \Sigma_{A'B'}).$$

(c) Work with the torsion-free $gl(2, \mathbb{C}) \oplus \widetilde{gl}(2, \mathbb{C})$ connection

$$\mathbb{G}_{AB} = \Gamma_{AB} + a\varepsilon_{AB} d\phi, \quad \mathbb{G}_{A'B'} = \Gamma_{A'B'} + (1-a)\varepsilon_{A'B'} d\phi$$

with $\Gamma_{AB} = \Gamma_{(AB)} \in sl(2, \mathbb{C}) \otimes \Lambda^1(T^*\mathcal{M})$, $\Gamma_{A'B'} = \Gamma_{(A'B')} \in \widetilde{sl}(2, \mathbb{C}) \otimes \Lambda^1(T^*\mathcal{M})$ and $a \in \mathbb{C}$. This leads to

$$d\hat{e}^a + \mathbb{G}^a{}_b \wedge \hat{e}^b = 0,$$

where $\mathbb{G}_{ab} = \Gamma_{ab} + \varepsilon_{A'B'}\varepsilon_{AB} d\phi$. The structure group reduces to

$$sl(2, \mathbb{C}) \oplus \widetilde{sl}(2, \mathbb{C}) \oplus u(1) \subset gl(2, \mathbb{C}) \oplus \widetilde{gl}(2, \mathbb{C}).$$

For (complexified) hyper-Hermitian four-manifolds $d\phi$ is replaced by the Lee form $-A$ in the above formulae. The possibility (a) would then correspond to the heterotic geometries studied by physicists in connection with $(4,0)$ supersymmetric σ-models (see [2] and references therein). Choice (b) is what we have used in this paper. Let us make a few remarks about the possibility (c).

Eq. (1) implies that $a = 1/2$ and

$$\mathbb{G}_{AB} = \Gamma_{AB} - 1/2\varepsilon_{AB} A, \quad \mathbb{G}_{A'B'} = -1/2\varepsilon_{A'B'} A$$

with $\Gamma_{AB} = \Gamma_{(AB)} \in sl(2, \mathbb{C})$. In the adopted coordinate system

$$\Gamma_{AA'BC} = -o_{A'} \left(\nabla_{(A0''}\nabla_{B0'}\Theta_{C)} + \frac{1}{2}\varepsilon_{BC} \frac{\partial\Theta^D}{\partial x^A \partial x^D} \right),$$

$$\Gamma_{AA'B'C'} = -\frac{1}{2} o_{A'}\varepsilon_{B'C'} \frac{\partial\Theta^D}{\partial x^A \partial x^D}.$$

The curvatures of $\mathbb{G}_{AB}$ and $\mathbb{G}_{A'B'}$ are

$$\mathbb{R}^A{}_B = d\mathbb{G}^A{}_B + \mathbb{G}^A{}_C \wedge \mathbb{G}^C{}_B = R^A{}_B - 1/2\varepsilon^A{}_B F, \quad \mathbb{R}^{A'}{}_{B'} = -1/2\varepsilon^{A'}{}_{B'} F,$$

where $F = dA$ is an ASD two-form. It would be interesting to investigate this possibility with connection to $gl(2, \mathbb{C})$ formulation of Einstein–Maxwell equations [15], and its Lagrangian description [16].

5.3. Reductions

Hyper-Hermitian four-manifolds which admit a tri-holomorphic vector field were recently studied in [2,3]. It would be interesting to look at the case of a general Killing vector taking Eq. (9) as a starting point. One might also consider reduction of real slices with $(++--)$ signature to obtain an 'evolution' form of Einstein–Weyl equations for metrics of signature $(+--)$.

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Appendix A

We shall use the conventions of Penrose and Rindler [12]: $a, b, \ldots$ are four-dimensional space–time indices and $A, B, \ldots, A', B', \ldots$ are two-dimensional spinor indices. The tangent space at each point of $\mathcal{M}$ is isomorphic to a tensor product of two spin spaces

$$T^a\mathcal{M} = S^A \otimes S^{A'}.$$

Spin dyads (o^A, ι^A) and $(o^{A'}, \iota^{A'})$ span S^A and $S^{A'}$, respectively. The spin spaces S^A and $S^{A'}$ are equipped with symplectic forms ε_{AB} and $\varepsilon_{A'B'}$ such that $\varepsilon_{01} = \varepsilon_{0'1'} = 1$. These anti-symmetric objects are used to raise and lower the spinor indices. We shall use the normalised spin frames, which implies that

$$o^B\iota^C - \iota^B o^C = \varepsilon^{BC}, \quad o^{B'}\iota^{C'} - \iota^{B'}o^{C'} = \varepsilon^{B'C'}.$$

Let $e^{AA'}$ be the null tetrad of one-forms on $\mathcal{M}$ and let $\nabla_{AA'}$ be the frame of dual vector fields. The orientation is fixed by setting $\nu = e^{01'} \wedge e^{10'} \wedge e^{11'} \wedge e^{00'}$. The local basis Σ^{AB} and $\Sigma^{A'B'}$ of spaces of ASD and SD two-forms are defined by

$$e^{AA'} \wedge e^{BB'} = \varepsilon^{AB}\Sigma^{A'B'} + \varepsilon^{A'B'}\Sigma^{AB}.$$

The Weyl tensor decomposes into ASD and SD part

$$C_{abcd} = \varepsilon_{A'B'}\varepsilon_{C'D'}C_{ABCD} + \varepsilon_{AB}\varepsilon_{CD}C_{A'B'C'D'}.$$

The first Cartan structure equations are

$$de^{AA'} = e^{BA'} \wedge \Gamma^A{}_B + e^{AB'} \wedge \Gamma^{A'}{}_{B'},$$

where Γ_{AB} and $\Gamma_{A'B'}$ are the $SL(2, \mathbb{C})$ and $\widetilde{SL}(2, \mathbb{C})$ spin connection one-forms symmetric in their indices, and

$$\Gamma_{AB} = \Gamma_{CC'AB}e^{CC'}, \quad \Gamma_{A'B'} = \Gamma_{CC'A'B'}e^{CC'},$$

$$\Gamma_{CC'A'B'} = o_{A'}\nabla_{CC'}\iota_{B'} - \iota_{A'}\nabla_{CC'}o_{B'}.$$

The curvature of the spin connection

$$R^A{}_B = d\Gamma^A{}_B + \Gamma^A{}_C \wedge \Gamma^C{}_B$$

decomposes as

$$R^A{}_B = C^A{}_{BCD}\Sigma^{CD} + (1/12)R\Sigma^A{}_B + \Phi^A{}_{BC'D'}\Sigma^{C'D'}$$

and similarly for $R^{A'}{}_{B'}$. Here R is the Ricci scalar and $\Phi_{ABA'B'}$ is the trace-free part of the Ricci tensor.

For convenience we express various spinor objects on $\mathcal{M}$ in terms of Θ_A.

tetrad	$e^{A0'} = dx^A + \dfrac{\partial\Theta^A}{\partial x^B}dw^B, \quad e^{A1'} = dw^A,$
dual tetrad	$\nabla_{A0'} = \dfrac{\partial}{\partial x^A}, \quad \nabla_{A1'} = \dfrac{\partial}{\partial w^A} - \dfrac{\partial\Theta^B}{\partial x^A}\dfrac{\partial}{\partial x^B},$
metric determinant	$\det(g) = 1$
Weyl spinors	$C_{A'B'D'E'} = 0, \quad C_{ABCD} = \nabla_{(A0'}\nabla_{B0'}\nabla_{C0'}\Theta_{D)},$
spin connections	$\Gamma_{AA'BC} = -\dfrac{1}{2}o_{A'}(\nabla_{(B0'}\nabla_{C0'}\Theta_{A)} + \nabla_{B0'}\nabla_{C0'}\Theta_A),$
	$\Gamma_{AA'B'C'} = -\dfrac{\partial^2\Theta^B}{\partial x^B\partial x^A}o_{(B'}\varepsilon_{C')A'},$
Lee form	$A = \dfrac{\partial^2\Theta^B}{\partial x^B\partial x^A}dw^A,$
wave operator	$\Box_g = A^a\partial_a + \nabla_{1'}^A\nabla_{A0'} = \dfrac{\partial^2}{\partial x_A\partial w^A} + \dfrac{\partial^2\Theta_B}{\partial x_A\partial x_B}\dfrac{\partial}{\partial x^A} + \dfrac{\partial\Theta^A}{\partial x_B}\dfrac{\partial}{\partial x^A}\dfrac{\partial}{\partial x^B},$
Ricci scalar	$R = 1/12(\nabla^a A_a + A_a A^a) = 0.$

The last formula follows because A is null and satisfies the Gauduchon gauge.

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